QR-submanifolds and Riemannian metrics with G_2 holonomy

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Abstract

In this note we prove that QR-submanifolds of the hyper-Kähler manifolds under some conditions admit the G_2 holonomy. We give simplest examples of such QR-submanifolds namely tori.

We conjecture that all G_2 holonomy manifolds arise in this way.

1 Introduction

The study of G_2-manifolds lacks explicit examples of closed manifolds. First complete Riemannian metrics with holonomy G_2 are constructed by Bryant and Salamon in [1]. First compact examples are given by Joyce in [2, 3]. Later Kovalev constructs more compact examples in [4, 5]. Note that metrics constructed in [2, 3, 4, 5] are not explicit.

Lack of examples is a consequence of the fact that G_2-manifolds are not generally algebraic in the broad sense of the term.

In this paper we try to partially explain this fact and conjecture that G_2-manifolds are generally QR-submanifolds of hyper-Kähler manifolds. Roughly speaking, QR-submanifolds are real hypersurfaces of hyper-Kähler manifolds.

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2 Preliminaries

2.1 G_2-structure

Define a 3-form Ω_0 on $\mathbb{R}^7$ by

$$\Omega_0 = x^{127} + x^{136} + x^{145} + x^{235} - x^{246} + x^{347} + x^{567}. \tag{1}$$

By x^{ijk} denote the $x^i \wedge x^j \wedge x^k$. The subgroup of $GL(7, \mathbb{R})$ preserving Ω_0 and orientation is called the G_2 group.

Let M be an oriented closed 7-manifold. Suppose there exists a global 3-form Ω such that pointwise it coincides with Ω_0; then M is called a G_2-manifold or we say that M carries the G_2-structure. It is known that the orientation and the Riemannian metric are uniquely determined by the G_2-structure.

2.2 Cross products

Let M be a G_2-manifold. Suppose a multilinear alternating smooth map $P : TM \times TM \to TM$. Suppose P satisfies compatibility conditions:

$$g(P(e_1, e_2), e_i) = 0, \quad i = 1, 2; \tag{2}$$

$$\|P(e_1, e_2)\|^2 = \|e_1\|^2\|e_2\|^2 - g(e_1, e_2)^2, \quad \|e\|^2 = g(e, e). \tag{3}$$

Then P is called a cross product. We also denote $P(e_1, e_2)$ by $e_1 \times e_2$.

The cross product is uniquely determined by the 3-form Ω:

$$\Omega(e_1, e_2, e_3) = g(P(e_1, e_2), e_3). \tag{4}$$

Conversely, the cross product defines the metric by the following formula:

$$P(e_1, P(e_1, e_2)) = -\|e_1\|^2 e_2 + g(e_1, e_2)e_1. \tag{5}$$

Using (4), we determine the 3-form Ω from the cross product and the metric. Thus the cross product implies the G_2-structure and vice versa.

Recall that if cross product is parallel with respect to the metric connection, then the holonomy group of M is a subgroup of G_2 and coincides with G_2 iff $\pi_1(M)$ is a finite group [2].

2.3 QR-submanifolds

Riemannian $4n$-manifold with holonomy group contained in $Sp(n)$ is called a hyper-Kähler manifold.

Suppose M is a submanifold of the hyper-Kähler $\overline{M}$ such that normal bundle of M is the direct sum of ν and $\nu^{\perp}$ and

$$J_i\nu \subset \nu, \quad J_i\nu^{\perp} \subset TM, \quad i = 1, 2, 3, \tag{6}$$

where by J_i we denote the ith complex structure of $\overline{M}$. Then M is called a QR-submanifold of $\overline{M}$.

In what follows we consider QR-submanifolds with $\dim \nu^{\perp} = 1$ only. We call them QR-submanifolds of the hypersurface type.

3 The main result

Theorem 1. *Let M be an oriented 7-manifold. If M is a hypersurface type QR-submanifold of hyper-Kähler $\overline{M}$, then there exists the G_2-structure on M.*

Proof. We shall construct a cross product on M such that it is compatible with the induced metric.

By (6), it follows that $\xi_i = J_i n$ are 3 non-vanishing vector fields on M. This agrees with [8], where existence of two non-vanishing vector fields on arbitrary compact orientable 7-manifold was shown. Third non-vanishing vector is the cross product of the first two (see also [9]).

We may assume that ξ_i are unit orthogonal with respect to the induced metric vector fields on M. Locally we extend ξ_i to a basis. Additional vectors are denoted by ξ_α, i.e., by Greek indices.

Let the cross product P be given by the following formulae:

$$P(\xi_i, \xi_j) = \xi_k, \quad (ijk) \in (123), \tag{7}$$
$$P(\xi_i, \xi_\alpha) = J_i(\xi_\alpha), \tag{8}$$
$$P(\xi_\alpha, J_i(\xi_\alpha)) = \xi_i. \tag{9}$$

By the definition of a hypersurface type QR-submanifold, we have that for any ξ_α, ξ_β there exists complex structure J_i such that $J_i\xi_\alpha = \xi_\beta$. Hence formulae (7)–(9) define the cross product on all basis vectors.

Clearly, P satisfies (5) and therefore P is compatible with the induced metric. ∎

Let's find out when the constructed cross product is parallel that is when holonomy is reduced to a subgroup of G_2.

Let $\overline{\nabla}$ and ∇ be a metric connection on $\overline{M}$ and M respectively.

Claim 1.

$$\begin{aligned} \nabla\xi_i &= J_i(\overline{\nabla}n) - b(\xi_i). && (10)\\ (\nabla J_i)(\xi_\alpha) &= J_i \circ b(\xi_\alpha) - b \circ J_i(\xi_\alpha). && (11) \end{aligned}$$

Proof. By the Gauss formula, we have

$$\overline{\nabla}\xi_i = \nabla\xi_i + b(\xi_i), \tag{12}$$

where $b(\xi_i) = b(\xi_i, \cdot)$ and b is the second fundamental form.

Also, the definition of the hyper-Kähler manifold implies that

$$\overline{\nabla}\xi_i = \overline{\nabla}J_i(n) = (\overline{\nabla}J_i)(n) + J_i(\overline{\nabla}n) = J_i(\overline{\nabla}n). \tag{13}$$

Combining (12) and (13), we get (10).

Similarly, combining

$$\overline{\nabla}(J_i\xi_\alpha) = \nabla(J_i(\xi_\alpha)) + b(J_i(\xi_\alpha)) = (\nabla J_i)(\xi_\alpha) + J_i(\nabla\xi_\alpha) + b(J_i(\xi_\alpha)) \tag{14}$$

and

$$\overline{\nabla}(J_i\xi_\alpha) = (\overline{\nabla}J_i)(\xi_\alpha) + J_i(\overline{\nabla}\xi_\alpha) = J_i(\overline{\nabla}\xi_\alpha), \tag{15}$$

we have (11). □

By definition, put

$$X_i(\xi) = J_i(\overline{\nabla}_\xi n) - b(J_i n, \xi), \quad Y_i(\xi,\eta) = J_i b(\xi,\eta) - b(\xi, J_i\eta).$$

Claim 2.

$$\begin{aligned} (\nabla P)(\xi_i,\xi_j) &= X_k - X_i \times \xi_j - \xi_i \times X_j. && (16)\\ (\nabla P)(\xi_i,\xi_\alpha) &= Y_i(\xi_\alpha) - X_i \times \xi_\alpha. && (17)\\ (\nabla P)(\xi_i,\xi_\alpha) &= Y_i(\xi_\alpha) - X_i \times \xi_\alpha. && (18) \end{aligned}$$

Proof. Let's prove (16). We differentiate (7):

$$(\nabla P)(\xi_i,\xi_j) = \nabla\xi_k - P(\nabla\xi_i,\xi_j) - P(\xi_i,\nabla\xi_j). \tag{19}$$

Combining (10), (19) and (7), we obtain (16).

Similarly, if we differentiate (8) and (9), we get (17) and (18).

□

Recall that $\nabla P = 0$ implies that $\mathrm{Hol}(M) \subset G_2$. If we equate with zero formulae (16)–(18), then we obtain sufficient conditions for $\nabla P = 0$. Note that (17) and (18) are equivalent.

Theorem 2. *Suppose M is an oriented 7-manifold such that M is a hypersurface type QR-submanifold of the hyper-Kähler $\overline{M}$. If the following equations hold:*

$$X_k(\xi) - X_i(\xi) \times \xi_j - \xi_i \times X_j(\xi) = 0, \tag{20}$$

$$Y_i(\xi, \eta) - X_i(\xi) \times \eta = 0, \tag{21}$$

for any $\xi, \eta, J_i\eta \in \Gamma(TM)$, $i = 1, 2, 3$; then holonomy group of M is contained in G_2.

Example. Simplest examples of QR-submanifolds with holonomy contained in G_2 are totally geodesic hypersurfaces. These are flat tori: $T^7 \hookrightarrow T^8$ and $T^3 \times K3 \hookrightarrow T^4 \times K3$.

4 Conjecture

Emery Thomas proves in [8] that any G_2-manifold admits 3 non-vanishing unit vector fields ξ_i. There exists a complex structure on $\xi_i^\perp$ determined by (5). Verbitsky shows in [10] that these complex structures are integrable iff the holonomy is contained in G_2. Due to integrability we formulate the following

Conjecture. *Any G_2 holonomy manifold is a QR-submanifold of a certain hyper-Kähler manifold satisfying the conditions of Theorem 2.*

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